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Brute force method for solving Ernst's equation and limits of the Kinnersley–Chitre solution

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Abstract. The programme POLYNOM is used to calculate a new solution of Ernst's equation. We then show that the solution is an extreme limit of the so-called Kinnersley–Chitre solution.

1. Introduction

Since the development of the various methods for generating solutions of Ernst's equation (cf e.g. Cosgrove 1980) the 'brute force' method has slightly fallen into disrepute. One makes an ansatz, assumes a certain form for the Ernst potential, plugs that into the equation and tries not to lose track of the various equations or constraints which emerge.

In this way Tomimatsu and Sato guided by experience gained from studies of approximate solutions of Ernst's equation (Sato and Tomimatsu 1973), discovered their solution (Tomimatsu and Sato 1973). Ernst, too, derived a series of unfortunately not asymptotically flat solutions (Ernst 1977).

The main obstacle for the application of the brute force method has been the sheer length of the necessary calculations. However, we have the programme POLYNOM at our disposal and we can use it to calculate the right-hand side of Ernst's equation for a specific ansatz.

This will be done in § 2. A new solution will be discovered. Section 3 shows that this solution is in fact an extreme case of the so-called Kinnersley–Chitre (1978) solution and it will be rederived in a way similar to the extreme limit of the Kerr solution.

2. Brute force method

Consider the Ernst equation

$$(\xi\xi^* - 1)\nabla_3^2\xi = 2\xi^*\nabla\xi^2 \tag{2.1}$$

and make the usual ansatz that ξ is a rational function

$$\xi = \beta\alpha^{-1}. \tag{2.2}$$

α and β are assumed to be polynomials in prolate spheroidal coordinates x, y

($\rho=[(x^2-1)(1-y^2)]^{1/2}$, $z=xy$) or polar coordinates r, $\cos\theta$ ($\rho=r\sin\theta$, $z=r\cos\theta$). It can be shown that for ξ which describe asymptotically flat solutions

$$\xi(-x,-y)=-\xi(x,y), \qquad \xi(-r,-\cos\theta)=-\xi(r,\cos\theta).$$

Hence α and β are sums of homogeneous polynomials of even, respectively odd, degree in x, y or r, $\cos\theta$.

Inserting the ansatz (2.2) into the Ernst equation one obtains

$$(\alpha\alpha^*-\beta\beta^*)(\beta\nabla_3^2\alpha-\alpha\nabla_3^2\beta)-2\{\alpha^*\beta\nabla\alpha^2+\beta^*\alpha\nabla\beta^2-(\alpha\alpha^*+\beta\beta^*)\nabla\alpha\nabla\beta\}=\mathrm{RHS}\overset{!}{=}0. \tag{2.3}$$

For α and β to constitute a solution, the right-hand side RHS has to vanish. The operators are

$$\nabla_3^2 f=[\partial_x(x^2-1)\partial_x+\partial_y(1-y^2)\partial_y]f, \qquad \nabla f\nabla g=(x^2-1)\partial_x f\partial_x g+(1-y^2)\partial_y f\partial_y g,$$

respectively

$$\nabla_3^2 f=(\sin\theta\partial_r r^2\partial_r+\partial_\theta\sin\theta\,\partial_\theta)f, \qquad \nabla f\nabla g=\sin\theta(r^2\partial_r f\,\partial_r g+\partial_\theta f\partial_\theta g).$$

Incidentially it may be remarked that all the substitutions $x\leftrightarrow y$, $x\to -x$, $y\to -y$, $r\to -r$, $\cos\theta\to -\cos\theta$ map solutions into solutions. Furthermore it should be noted that if one assumes the leading homogeneous polynomials in α and β to be of orders δ and $\delta-1$, the leading one of RHS is of order $4\delta-3$. The polynomial of order $4\delta-1$ in RHS vanishes identically.

Now expand α and β in x respectively r. The Laplacian in prolate spheroidals changes the order in x, while the Laplacian in polar coordinates leaves the order of the r expansion intact.

Therefore we shall in the following use polar coordinates.

It is easy to see that the r^0 term in RHS contains only contributions from the r^0 terms in α and β. This means that the r^0 terms in α and β have to be r-independent solutions of Ernst's equation, namely

$$\alpha=f(\theta)A_+^\delta, \qquad \beta=f(\theta)A_-^\delta, \qquad A_\pm^\delta=\tfrac{1}{2}[(\cos\theta+1)^\delta\pm(\cos\theta-1)^\delta],$$

with some (so far) arbitrary function $f(\theta)$. From the already published solution (Hoenselaers 1981) we observe, however, that the terms which contain the highest exponents of the appearing parameters are of the form

$$\sin^{\delta 2-\delta}\theta A_\pm^\delta$$

if α is a polynomial of order δ^2.

If we restrict ourselves to $\delta=2$ and furthermore assume the solution to be symmetric with respect to the equatorial plane $\theta=\pi/2$, i.e.

$$\alpha(r,-\cos\theta)=\alpha^*(r,\cos\theta), \qquad \beta(r,-\cos\theta)=\beta^*(r,\cos\theta), \tag{2.4}$$

we can make the ansatz

$$\beta=a\mathrm{i}\sin^2\theta A_-^2+br\sin^2\theta+cr, \qquad \alpha=a\sin^2\theta A_+^2+\mathrm{i}\cos\theta\,(dr\sin^2\theta+er),$$

where, at present, we are only interested in the r^1 term of RHS. Higher r terms in α and β do not contribute. a, b, etc are constants. We find the conditions

$$c+e=0, \qquad 4b+5c-2d=0.$$

Now we modify the ansatz for α and β

$$\beta = a\,\mathrm{i}\sin^2\theta\, A_-^2 + r(b\sin^2\theta + 2c) + \mathrm{i}d\cos\theta\, r^2,$$

$$\alpha = a\sin^2\theta\, A_+^2 + \mathrm{i}r\cos\theta[(2b+5c)\sin^2\theta - 2c] + r^2(e\sin\theta + f),$$

calculate the r^2 term of RHS and take account of the emerging relations among the constants.

It is now clear how this process can be repeated. The reason for choosing the iterative method instead of simply inserting the most general ansatz under our assumptions into the Ernst equation and then examining the RHS is that the Ansatz would contain ten constants and the RHS would run up to about 1200 terms.

In any case, at the end of the calculation there remains a solution depending on two parameters

$$\alpha = r^4 + 2ab\cos^2\theta\, r^2 - ab^3\sin^2\theta\,(1+\cos^2\theta) + \mathrm{i}r\cos\theta\,[(a-b)r^2 - 4ab^2\sin^2\theta], \tag{2.5a}$$

$$\beta = (a+b)r^3 - 2ab^2\sin^2\theta\, r - 2\mathrm{i}\cos\theta\, ab(r^2 + \sin^2\theta\, b^2). \tag{2.5b}$$

If we set either a or b to zero, the solution reduces to the extreme Kerr solution. Further details about this solution will be published elsewhere (Hoenselaers and Heß 1982).

3. Limits of the Kinnersley–Chitre solution

By application of the so-called β_k transformations to the $\delta = 2$ Voorhees solution, Kinnersley and Chitre (1978) derived the following solution of Ernst's equation:

$$\alpha = p^2(x^4-1) - 2\mathrm{i}pqxy(x^2-y^2) + q^2(y^4-1) - 2\mathrm{i}\lambda(x^2+y^2-2x^2y^2)$$
$$\quad - 2\mathrm{i}\mu xy(x^2+y^2-2) + (\lambda^2-\mu^2)(x^2-y^2)^2, \tag{3.1a}$$

$$\beta = 2px(x^2-1) - 2\mathrm{i}qy(1-y^2) - 2\mathrm{i}(x^2-y^2)[x(p\lambda + \mathrm{i}q\mu) - y(p\mu + \mathrm{i}q\lambda)], \tag{3.1b}$$

$$p^2 + q^2 = 1.$$

For $\lambda = \mu = 0$ this reduces to the $\delta = 2$ Tomimatsu–Sato solution. For $q = 0$ respectively $p = 0$ it reduces to the solutions quoted in Hoenselaers *et al* (1979b). The full metric for this solution has been given by Yamazaki (1980). Kinnersley and Kelley (1974) have shown how to take the $p \to 0$ limit of the Tomimatsu–Sato solutions while keeping px^n finite for odd n. $n = 1$ always yielded the extreme Kerr solution, while other n gave new, not asymptotically flat solutions.

Here we shall investigate the limit $x \to \infty$ and show how the parameters have to be chosen for the limit to yield a finite result. We replace

$$x \to r/r_0\varepsilon, \qquad y = \cos\theta, \tag{3.2}$$

and thereby introduce a dimensionless parameter ε. Before taking the limit $\varepsilon \to 0$ we have to keep in mind that we can multiply α and β by ε^n $(n = 0 \ldots 3)$.

Let us first examine the case $n = 3$. The only term multiplied by ε^{-1} in this case is the x^4 term in α. Hence

$$p^2 + \lambda^2 - \mu^2 \simeq \varepsilon.$$

The symbol $\simeq \varepsilon^k$ means that the expression has to go to zero at least as fast as ε^k. If we thus set, for instance, $p = \cos\phi + \varepsilon p_1$, $\lambda = \cos\phi \cosh\psi + \varepsilon\lambda_1$, $\mu = \cos\phi \sinh\psi + \varepsilon\mu_1$ and let $\varepsilon \to 0$, we recover the extreme Kerr solution.

Detailed calculation shows also that nothing interesting comes up for $n = 1, 2$. For $n = 0$, however, we have the following order equations:

$$\begin{gathered} p^2+\lambda^2-\mu^2 \simeq \varepsilon^4, \qquad \mu + pq \simeq \varepsilon^3, \qquad \mu q + p \simeq \varepsilon^3, \qquad \lambda p \simeq \varepsilon^3, \\ \mu^2 - \lambda^2 \simeq \varepsilon^2, \qquad \lambda \simeq \varepsilon^3, \qquad p\mu \simeq \varepsilon^2, \qquad \lambda q \simeq \varepsilon^2, \\ \mu \simeq \varepsilon, \qquad pq \simeq \varepsilon, \qquad p \simeq \varepsilon, \qquad \mu q \simeq \varepsilon. \end{gathered} \tag{3.3}$$

The only consistent solution is

$$p \simeq p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3, \qquad \lambda \simeq \lambda_2\varepsilon^2, \qquad \mu \simeq -p_1\varepsilon - p_2\varepsilon^2 + \mu_3\varepsilon^3. \tag{3.4}$$

If we now let $\varepsilon \to 0$ we find

$$\begin{aligned} \alpha = {} & r^4[2p_1(p_3+\mu_3)+\lambda_2^2] + 2\cos^2\theta\, r^2 p_1^2 - \sin^2\theta\,(1+\cos^2\theta) \\ & + \mathrm{i}\{\cos\theta\, r^3[p^3 - 2(p_3+\mu_3)] + 2\lambda_2 r^2(\cos^2\theta - \sin^2\theta) - 4\sin^2\theta\cos\theta\, p_1\}, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} \beta = {} & r^3[2(p_3+\mu_3)+6p_1^3] - 2\cos\theta\, r^2\lambda_2 - 2\sin^2\theta\, p_1 r \\ & - \mathrm{i}(2\lambda_2 p_1 r^3 + 2p_1\cos\theta\, r^2 + 2\sin^2\theta\cos\theta). \end{aligned} \tag{3.5b}$$

r_0 has been absorbed into the other constants. It is obvious that $p_3+\mu_3$ can be replaced by a single constant.

For $p_1 = 0$ we recover the solution obtained from a HKX rank-1 transformation applied to flat space (Hoenselaers *et al* 1979a). If $\lambda_2^2 = -2p_1(p_3+\mu_3)$ the solution will not be asymptotically flat and is actually a two-parameter generalisation of the solutions derived in Kinnersley and Kelley (1974). For $\lambda_2 = 0$ we arrive at the solution of § 2.

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